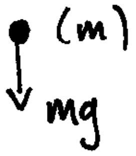
SOLUTIONS 4

Q1. (i) ⊕ ↓ increasing z. Newton's 2nd law : $m\ddot{z} = mg$.

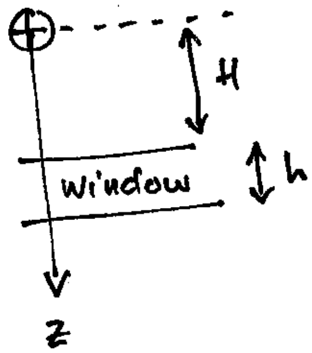


So $\ddot{z} = g$.

$$\Rightarrow \dot{z} = gt + c \quad , \quad z = \frac{gt^2}{2} + ct + d .$$

$z = 0$ & $\dot{z} = 0$ at $t = 0$ $\Rightarrow$ $\dot{z} = gt$ & $\boxed{z = \frac{gt^2}{2}.}$ $(c, d = 0)$

(ii) We want to find H.



$h = 1\,m \qquad \tau = 0.05\,s.$

when $z = H$, $t = T$

$z = H + h$, $t = T + \tau$

$$\text{So } H = \frac{gT^2}{2} \quad , \quad H + h = \frac{g(T+\tau)^2}{2}$$

$$\Rightarrow \cancel{H} + h = \frac{g}{2}\left(\cancel{T^2} + 2T\tau + \tau^2\right)$$

$$h = gT\tau + \frac{g\tau^2}{2} \Rightarrow T = \frac{1}{g\tau}\left(h - \frac{g\tau^2}{2}\right)$$

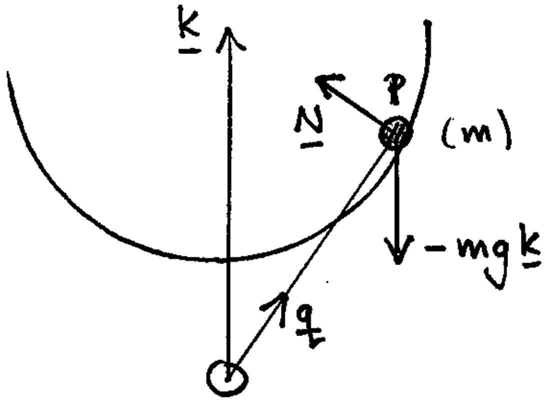
$$\text{then} \quad H = \frac{g}{2}\left[\frac{1}{g\tau}\left(h - \frac{g\tau^2}{2}\right)\right]^2$$

$$= \frac{g}{2}\left(\frac{h}{g\tau} - \frac{\tau}{2}\right)^2 \simeq 19.9\,m$$

ie. from the 15th floor.

{ One could also make an approximation of $\dot{z} \simeq h/\tau$ $(= 20\,m/s)$ at the window instead!

Q2



Only forces acting on P are the weight & normal reaction $\underline{N}$.

(i) The moment about O is $\underline{L}_O = \underline{q} \wedge \underline{F} = \underline{q} \wedge (\underline{N} - mg\underline{k})$

$$\Rightarrow \underline{L}_O = \underline{q} \wedge \underline{N} - \underline{q} \wedge mg\underline{k}$$

The vertical component is then $\underline{L}_O \cdot \underline{k}$,

$$\underline{L}_O \cdot \underline{k} = \underbrace{(\underline{q} \wedge \underline{N}) \cdot \underline{k}}_{=0 \text{ since } \underline{q}, \underline{N} \,\&\, \underline{k} \text{ are co-planar.}} - mg \overbrace{(\underline{q} \wedge \underline{k}) \cdot \underline{k}}^{=0} = 0$$

(ii) From $\frac{d}{dt}\underline{H}_O = \underline{L}_O$, $\quad \dot{\underline{H}}_O \cdot \underline{k} = \underline{L}_O \cdot \underline{k} = 0$

So $\frac{d}{dt}(\underline{H}_O \cdot \underline{k}) = 0 \Rightarrow \underline{H}_O \cdot \underline{k} = \text{constant} = h$ say.

(iii) $\underline{H}_O = \underline{q} \wedge m\dot{\underline{q}}$, $\quad \underline{q} = r\hat{\underline{r}} + z\underline{k}$, $\quad \dot{\underline{q}} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}} + \dot{z}\underline{k}$

$\underline{H}_O \cdot \underline{k}$ is the $\underline{k}$ component of $\underline{H}_O$

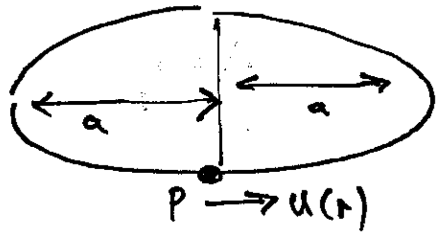
the only terms that give a $\underline{k}$ in the cross product are $\hat{\underline{r}} \wedge \hat{\underline{\theta}} = \underline{k}$ (other terms are: $\hat{\underline{r}} \wedge \underline{k} = -\hat{\underline{\theta}}$ & $\underline{k} \wedge \hat{\underline{\theta}} = -\hat{\underline{r}}$).

So $(\underline{q} \wedge m\dot{\underline{q}}) \cdot \underline{k}$

$$= mr(r\dot{\theta}) = mr^2\dot{\theta} = h \text{ (a constant)}.$$

Q3 $T + V = E$ leads to (when $m = 1$)

$$\frac{1}{2}[u(r)]^2 - \frac{MG}{r} = -\frac{MG}{2a} \qquad \text{——— (I)}$$



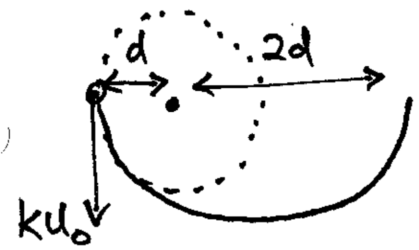
} elliptical orbit of semi-major axis a & speed $u(r)$, where r is the distance from centre of the Earth.

(i) Orbit is a circle $r = a = d$ & speed is constant

$u(r) \equiv u_0$.

$$(\mathrm{I}) \Rightarrow \frac{1}{2}u_0^2 - \frac{MG}{d} = -\frac{MG}{2d} \Rightarrow u_0^2 = \frac{MG}{d}.$$

(ii) Now we change the speed from $u_0 \to k u_0$.
We want to choose k s.t. semi-major axis becomes $3d/2$.



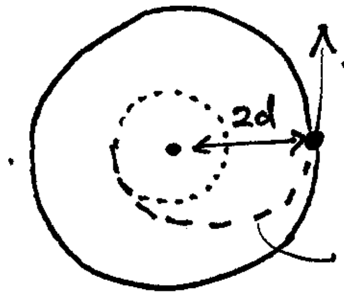
if $a = \frac{3d}{2}$, then we need an energy $-\frac{MG}{2a} = -\frac{MG}{3d} = E$ say

$$\text{So } (\mathrm{I}) \Rightarrow \quad \frac{1}{2}k^2 u_0^2 - \frac{MG}{d} = -\frac{MG}{3d}$$

$$\Rightarrow \quad \frac{1}{2}k^2 u_0^2 = \frac{2MG}{3d}$$

$$\text{So} \quad k^2 = \frac{4MG}{3d} \Big/ \frac{MG}{d} = \frac{4}{3}, \qquad k = \sqrt{4/3}$$

(iii) When $r = 2d$ we increase the speed from $u_1 \longrightarrow \mu u_1$. We want to choose μ s.t we have a circular orbit of radius $2d$.



First we need to find u_1. For the transfer orbit:

$$\frac{1}{2} u_1^2 - \frac{MG}{2d} = E = -\frac{MG}{3d}$$

$$\text{So} \quad \frac{u_1^2}{2} = \frac{MG}{6d} \qquad \therefore \; u_1^2 = \frac{MG}{3d}.$$

The energy required for the circular orbit of radius $2d$ is $-\dfrac{MG}{2.2d} = -\dfrac{MG}{4d}$.

$$\text{so (I)} \Rightarrow \quad \frac{1}{2}\mu^2 u_1^2 - \frac{MG}{2d} = -\frac{MG}{4d}$$

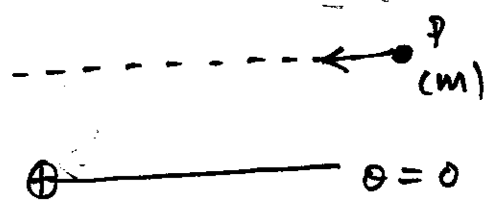
$$\frac{1}{2}\mu^2 u_1^2 = \frac{MG}{4d}$$

$$\mu^2 = \frac{MG}{2d} \Big/ u_1^2 = \frac{MG}{2d} \Big/ \left(\frac{MG}{3d}\right) = \frac{3}{2}$$

$$\text{so} \quad \mu = \sqrt{3/2}.$$

<u>Q4</u>

$$\underline{F} = -\frac{m\gamma^2}{r^3}\,\hat{\underline{r}} \quad \text{using } f(r) = -\frac{\gamma^2}{r^3}.$$



We are told that $H_0 = \dfrac{15\gamma}{\sqrt{209}}$

P is projected with $H_0 = \dfrac{15\gamma}{\sqrt{209}}$

path equation is:

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{H_0^2 u^2}$$

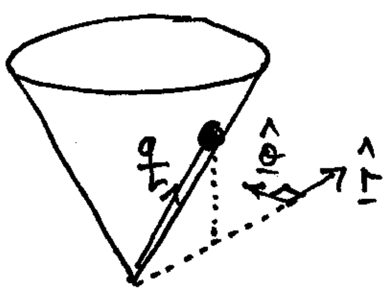
Here $f(1/u) = -\gamma^2 u^3$ so $-\dfrac{f(1/u)}{H_0^2 u^2} = \dfrac{\gamma^2 u}{H_0^2}$

$$\Rightarrow \quad \frac{d^2u}{d\theta^2} + u\left(1 - \frac{\gamma^2}{H_0^2}\right) = 0$$

$$u'' + u\left(1 - \frac{209}{15^2}\right) = 0$$

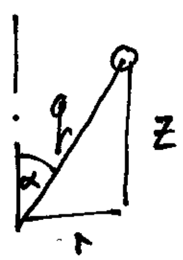
$$u'' + \frac{4^2}{15^2}u = 0, \quad u = \alpha\cos\frac{4\theta}{15} + \beta\sin\frac{4\theta}{15} = \frac{1}{r}.$$

Q5.



At $t=0$ $|\underline{q}| = a$
and $\dot{\underline{q}} = u\hat{\underline{\theta}}$

(i)



$r = q\sin\alpha \quad z = q\cos\alpha$

$Q2 \Rightarrow mr^2\dot{\theta} = h$ (const.)

at $t=0$ $|\underline{q}| = q = a$

and $\dot{\underline{q}} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}} + \dot{z}\underline{k} = u\hat{\underline{\theta}}$

$\Rightarrow \dot{r} = 0 \,,\; \dot{z} = 0 \quad \& \quad \underline{q} \sin\alpha \, \dot{\theta} = U \,.$

So $\quad m r^2 \dot{\theta} = m(\underline{q}\sin\alpha)^2 \dot{\theta} = m a \sin\alpha \, U = h$ (const)

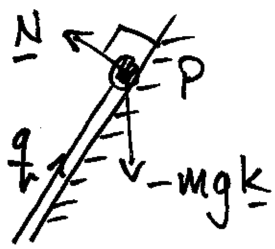
$\boxed{h = m a \sin\alpha \, U}$ constant for the motion.

At general t

$$m \underline{q}^2 \sin^2\alpha \, \dot{\theta} = h = m a \sin\alpha \, U$$

$$\Rightarrow \boxed{\underline{q}^2 \sin\alpha \, \dot{\theta} = a U .}$$

(ii) $\underline{F} = -mg\underline{k} + \underline{N}$



So $\underline{F} \cdot \dot{\underline{q}} = \dot{\underline{q}} \cdot \underline{N} - mg \, \dot{\underline{q}} \cdot \underline{k}$

Brief: Because P is on the cone & $\underline{N}$ is perp. to the cone $\quad \dot{\underline{q}} \cdot \underline{N} = 0$, so $\underline{F} \cdot \dot{\underline{q}} = -mg \, \dot{\underline{q}} \cdot \underline{k} = -mg\dot{z}$

or $\quad \underline{N} = N(-\cos\alpha \, \hat{\underline{r}} + \sin\alpha \, \underline{k})$ (as perp. to cone)

where $N = |\underline{N}|$, so

$$\underline{N} \cdot \dot{\underline{q}} = N \begin{pmatrix} -\cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix} \cdot \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ \dot{z} \end{pmatrix}$$

but as $r = \tan\alpha \, z \qquad \dot{r} = \tan\alpha \, \dot{z}$

So $\underline{N} \cdot \dot{\underset{\sim}{q}} = N\dot{z}\,(-\cos\alpha\tan\alpha + \sin\alpha) = 0.$

(iii) So the reaction force $\underline{N}$ does no work &

$$V = -\int \underline{F} \cdot \dot{\underset{\sim}{q}}\,dt = +\int mg\dot{z}\,dt = mgz + V_0$$

we can choose $V_0 = 0$.

(iv) Hence $T + V = E \Rightarrow \frac{1}{2}m|\dot{\underset{\sim}{q}}|^2 + mgz = E$

$$|\dot{\underset{\sim}{q}}|^2 = \dot{\underset{\sim}{q}} \cdot \dot{\underset{\sim}{q}} = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 = \dot{q}^2\sin^2\alpha + q^2\sin^2\alpha\,\dot{\theta}^2 + \dot{q}^2\cos^2\alpha$$

As usual, evaluate E using $t=0$ information.

<u>At $t=0$</u> $\quad \dot{\underset{\sim}{q}} = u\hat{\underline{\theta}} \quad |\dot{\underset{\sim}{q}}|^2 = u^2$ & $z = a\cos\alpha$

so $E = \frac{1}{2}mu^2 + mga\cos\alpha.$

<u>So for $t>0$</u>

$$\frac{1}{2}m(\dot{q}^2 + q^2\sin^2\alpha\,\dot{\theta}^2) + mgq\cos\alpha = \frac{1}{2}mu^2 + mga\cos\alpha$$

as usual, eliminate $\dot{\theta}$ using part (i) above $\dot{\theta} = \dfrac{au}{q^2\sin\alpha}$

$$\Rightarrow \frac{1}{2}\cancel{m}\left(\dot{q}^2 + \frac{a^2u^2}{q^2}\right) + \cancel{m}gq\cos\alpha = \frac{\cancel{m}u^2}{2} + \cancel{m}ga\cos\alpha$$

$$\Rightarrow \dot{q}^2 = (q-a)\left[\frac{u^2(q+a)}{q^2} - 2g\cos\alpha\right] \text{ as req}^{d}.$$

(v) Find the max/min distance of P from the apex of the cone $\Rightarrow$ we want $\dot{q} = 0$ at a max/min point.

$q = a$ is a solution (obvious from initial condition)

other solution requires

$$u^2(q+a) = 2g\,q^2\cos\alpha \quad \text{a quadratic for } q.$$

$$q^2(2g\cos\alpha) - q(u^2) - u^2 a = 0$$

$$\Rightarrow q = \frac{u^2 \pm \sqrt{u^4 + 4.2g\cos\alpha\, u^2 a}}{2.2g\cos\alpha}$$

$$= \frac{u^2 \pm u\sqrt{u^2 + 8ga\cos\alpha}}{2.2g\cos\alpha}$$

<u>Note</u>: $u^2 + 8ga\cos\alpha \geqslant u^2$ so we must take the +ve sign to get $q \geqslant 0$.

So $\dot{q} = 0$ when $q = a$ <u>or</u>

$$q = \frac{u^2 + u\sqrt{u^2 + 8ga\cos\alpha}}{4g\cos\alpha} = q^*, \text{ say}$$

note $q^* \to \infty$ as $g \to 0$ as one should expect!

Also, $q^* = a$ when $u^2 = ga\cos\alpha$, so at this value $\dot{q} = 0$ and P stays at the same level.