Solutions 1

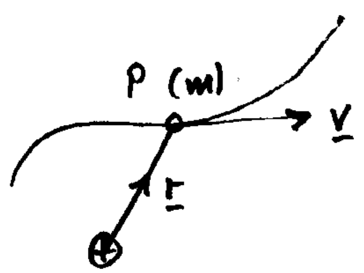
Q1. $\underline{r} = (2t^2 - 5t)\underline{i} + (4t+2)\underline{j} + t^3\underline{k}$.

So $\underline{v} = \dfrac{d\underline{r}}{dt} = \dot{\underline{r}} = (4t-5)\underline{i} + 4\underline{j} + 3t^2\underline{k}$ | velocity.

and $\underline{a} = \dfrac{d\underline{v}}{dt} = \ddot{\underline{r}} = 4\underline{i} + 6t\underline{k}$ | acceleration.

When $t = 2$ $\quad \underline{v} = (4.2-5)\underline{i} + 4\underline{j} + 3.2^2\underline{k} = 3\underline{i} + 4\underline{j} + 12\underline{k}$

so the speed is $|\underline{v}| = v = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13$.



the velocity is tangential to the path at any given time.

so a unit vector parallel to the velocity

is $\dfrac{\underline{v}}{|\underline{v}|} = \dfrac{1}{13}\left(3\underline{i} + 4\underline{j} + 12\underline{k}\right)$

The linear momentum is defined to be $\underline{p} = m\underline{v}$ (lecture #1)

So $\underline{p} = m(3\underline{i} + 4\underline{j} + 12\underline{k})$ at time $t = 2$

or $\underline{p} = m((4t-5)\underline{i} + 4\underline{j} + 3t^2\underline{k})$ generally.

Q2. The acceleration $\underline{a} = \ddot{\underline{r}} = 6t\,\underline{i} + 10\,\underline{k}$,

integrating: $\quad \dot{\underline{r}} = \dfrac{6t^2}{2}\underline{i} + 10t\,\underline{k} + \underline{c}$

where $\underline{c}$ is a (vector) constant of integration.

We are told that $\dot{\underline{r}}(0) = -2\underline{j} + \underline{k}$

so when $t=0$ $\quad -2\underline{j} + \underline{k} = 3.0^2\underline{i} + 10.0\,\underline{k} + \underline{c}$

$$\Rightarrow \underline{c} = -2\underline{j} + \underline{k}$$

then $\quad \underline{r} = \dfrac{3t^3}{3}\underline{i} + \dfrac{10t^2}{2}\underline{k} + (\underline{k} - 2\underline{j})t + \underline{d}$

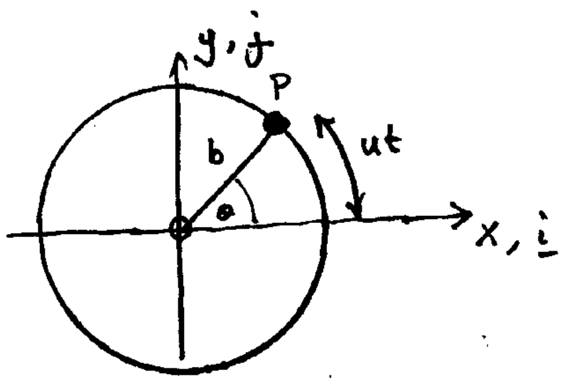
for some constant $\underline{d}$.

We are told that $\underline{r}(0) = \underline{0}$

so $\quad \underline{d} = \underline{0}$

and $\quad \underline{r} = t^3\underline{i} + (5t^2 + t)\underline{k} - 2t\underline{j}$.

Q3.



The speed is u, so after a time t, P is at (x,y) having travelled a distance ut.

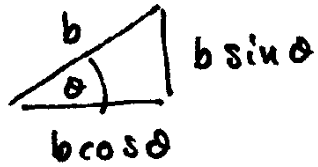
The arc length of a sector of angle θ is $b.\theta$ (ie radius x angle).

So $\quad ut = b\theta \quad \& \quad \theta = ut/b$.

The position vector of P is

$$\underline{r} = x\underline{i} + y\underline{j}$$

where ,



, $x = b\cos\theta$
$y = b\sin\theta$

So $\underline{r} = b\cos\theta\,\underline{i} + b\sin\theta\,\underline{j}$

$$= b\cos\left(\frac{ut}{b}\right)\underline{i} + b\sin\left(\frac{ut}{b}\right)\underline{j}$$

We are told in the question that $\omega = u/b$, then

$$\underline{r} = b\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}, \quad \dot{\underline{r}} = b\omega\begin{pmatrix}-\sin\omega t\\ \cos\omega t\end{pmatrix}$$

and $\ddot{\underline{r}} = -b\omega^2\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}$ simply by diff. w.r.t time.

$$\Rightarrow \ddot{\underline{r}} = -\omega^2\,\underline{r}.$$

Q4. We are given that $\underline{v} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}}$

The acceleration is $\underline{a} = \dfrac{d\underline{v}}{dt}$, & we use the "product rule"

So $\underline{a} = \ddot{r}\,\hat{\underline{r}} + \dot{r}\dfrac{d\hat{\underline{r}}}{dt} + \dfrac{d}{dt}(r\dot{\theta})\,\hat{\underline{\theta}} + r\dot{\theta}\dfrac{d\hat{\underline{\theta}}}{dt}$

As derived in the lectures $\dfrac{d\hat{\underline{r}}}{dt} = \dot{\theta}\,\hat{\underline{\theta}}$ & $\dfrac{d\hat{\underline{\theta}}}{dt} = -\dot{\theta}\,\hat{\underline{r}}$

So $\underline{a} = \ddot{r}\,\hat{\underline{r}} + \dot{r}\dot{\theta}\,\hat{\underline{\theta}} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}} + r\dot{\theta}(-\dot{\theta}\,\hat{\underline{r}})$

$$\Rightarrow \quad \underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\underline{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\underline{\theta}} \qquad \text{———} (*)$$

For constant speed motion around a circle of radius b (as in Q3):

$$\left.\begin{aligned} r = b \quad &\Rightarrow \quad \dot{r} = \ddot{r} = 0 \\ \theta = \frac{ut}{b} = \omega t \quad &\Rightarrow \quad \dot{\theta} = \omega, \ \ddot{\theta} = 0. \end{aligned}\right\}$$

- Subst. into $(*)$ $\Rightarrow$

$$\underline{a} = (0 - b.\omega^2)\,\hat{\underline{r}} + (b.0 + 2.0\,\omega)\,\hat{\underline{\theta}}$$

$$= -b\,\omega^2\,\hat{\underline{r}}$$

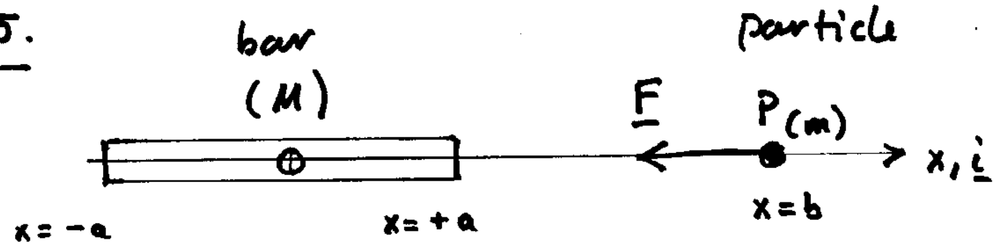
since $\underline{a} = \dfrac{d^2\underline{r}}{dt^2} = \ddot{\underline{r}}$, we have $\ddot{\underline{r}} = -b\omega^2\,\hat{\underline{r}}$

- but remember that $\underline{r} = r\,\hat{\underline{r}}$, where $\hat{\underline{r}}$ is the unit vector, so $\underline{r} = b\,\hat{\underline{r}}$

$$\Rightarrow \quad \ddot{\underline{r}} = -\cancel{b}\,\omega^2 . \frac{\underline{r}}{\cancel{b}}$$

So $\ddot{\underline{r}} = -\omega^2\,\underline{r}$ as in Q3.

Q5.

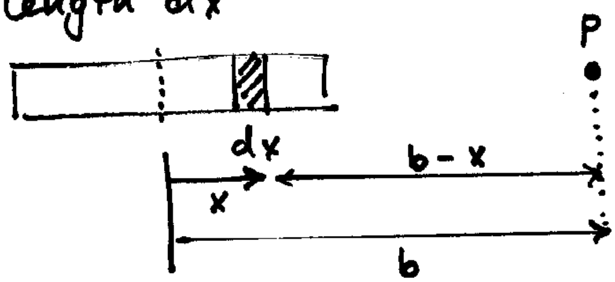


Since everything is aligned with the x-axis, $\underline{F} = -F(b)\,\underline{i}$

We wish to find $F(b)$, the force as a function of the particle's location.

We consider the attraction due to a small element of

- the bar of length dx



The element is at point x, so the separation distance from P is $b - x = R$ say.

- The mass of the element is

$$\text{mass of the bar} \times \left(\frac{\text{length of the element}}{\text{length of the bar}} \right)$$

$$= M \cdot \frac{dx}{2a} = M_e \text{ say.}$$

Newton's law of gravitation gives the force to be

$$\frac{m M_e G}{R^2} = \frac{m \left(\frac{M\,dx}{2a} \right) G}{(b-x)^2}$$

The <u>total</u> force is the attraction to <u>all</u> such elements in the bar, ie. we need to integrate over the length of the bar:

$$\Rightarrow \quad F(b) = \int_{-a}^{+a} \frac{mMG}{2a} \frac{1}{(b-x)^2}\,dx \qquad \left\{ \begin{array}{l} \text{subst. } s = b-x \\ ds = -dx \end{array} \right.$$

$$= \frac{mMG}{2a} \int_{b+a}^{b-a} -\frac{ds}{s^2} = \frac{mMG}{2a} \left[\frac{1}{s} \right]_{b+a}^{b-a}$$

$$= \frac{mMG}{2a} \left\{ \frac{1}{b-a} - \frac{1}{b+a} \right\} = \frac{mMG}{2a} \left\{ \frac{2a}{(b-a)(b+a)} \right\}$$

$$= \frac{mMG}{b^2 - a^2} \quad \leftarrow \text{the magnitude}$$

The <u>vector</u> force is $\underline{F} = -F(b)\,\underline{i} = -\dfrac{mMG}{b^2 - a^2}\,\underline{i}$

<u>Note</u>: if <u>$b \gg a$</u>, then we can treat the bar as a particle of mass M at O, and

$$\underline{F} \sim -\frac{mMG}{b^2}\,\underline{i}.$$