SECTION B

Answer **THREE** of the four questions

B5. A particle P, of mass m, is projected with an initial velocity $U_0\underline{k}$, where $U_0 > 0$ and $\underline{k}$ is a unit vector in the upwards vertical direction. P moves in a uniform gravitational field $-g\underline{k}$ in a medium that exerts a resistance force proportional to the fourth power of the particle's speed.

(i) If the velocity of P is $v(t)\underline{k}$, explain why the equation of motion for $v > 0$ can be written as

$$\frac{\mathrm{d}v}{\mathrm{d}t}\underline{k} = -g\underline{k} - \gamma v^4 \underline{k},$$

where $\gamma > 0$ is a constant of proportionality associated with the air resistance force.

(ii) Obtain an expression for the maximum height that P can reach above its projection point. Hence, show that the particle can never be projected any higher than $\pi/(4\sqrt{\gamma g})$ for any U_0.

Hint:
You may find the following standard integral helpful,

$$\int \frac{1}{1+s^2}\,\mathrm{d}s = \tan^{-1}(s),$$

when used with the substitution $s = v^2\sqrt{\gamma/g}$.

[13 marks]

B6. A particle P of unit mass moves along the positive x-axis under the influence of a force

$$F(x) = \frac{36}{x^3} - \frac{9}{x^2}, \quad x > 0.$$

(i) Sketch the shape of a potential function $V(x)$, where $V'(x) = -F(x)$. Show that there is a single equilibrium position for P and find the energy associated with this equilibrium state.

(ii) Initially P is projected from the equilibrium position with speed $1/2$. Use the conservation of energy equation to show that in the subsequent motion, the particle position is bounded between $x = 3$ and $x = 6$.

(iii) Express the time taken for the particle to travel from $x = 3$ to $x = 6$ as a definite integral with respect to x.

[13 marks]

B7. A system of particles $\{P_1, P_2, \ldots, P_N\}$ with position vectors $\{\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N\}$ relative to an origin O are acted upon by the system of forces $\{\underline{F}_1, \underline{F}_2, \ldots, \underline{F}_N\}$ respectively.

(i) State the definition of the resultant force, $\underline{F}$, and the total moment about the origin, $\underline{L}_O$, for this system.

(ii) Prove that when $\underline{F} = \underline{0}$, the total moment is the same about every point in space.

(iii) Suppose the system is in a uniform gravitational field, such that $\underline{F}_i = -m_i g \underline{k}$, for $i = 1, 2, \ldots, N$ where m_i is the mass of particle P_i, g is the constant local gravitational acceleration and $\underline{k}$ is a unit vector pointing vertically upwards. Show that the moment about the origin, $\underline{L}_O$, can be written as

$$\underline{L}_O = \underline{R} \wedge (-Mg\underline{k}),$$

where M is the total mass and

$$\underline{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \underline{r}_i .$$

(iv) Hence or otherwise, find the total moment about the origin, $\underline{L}_O$, for the case $N = 2$, where $m_1 = 2$, $m_2 = 1$, $\underline{r}_1 = \underline{i} + \underline{j}$ and $\underline{r}_2 = -\underline{i} + \underline{k}$; here $\{\underline{i}, \underline{j}, \underline{k}\}$ are the unit vectors of a Cartesian coordinate system centred at the origin.

[13 marks]

B8. A particle P of mass m moves in a central field of force of the form

$$\underline{F} = mf(r)\hat{\underline{r}} .$$

You are given that the motion of P satisfies the 'path equation'

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = -\frac{1}{H_0^2 u^2} f(1/u),$$

where $u = 1/r$, θ is the polar angle, and H_0 is a constant angular momentum such that $H_0^2 = -a^3 f(a)$ where a is a positive constant.

(i) Verify that a circular orbit, $u = 1/a$, is a solution of the path equation.

(ii) Now suppose that P is disturbed slightly from this circular orbit, with H_0 unchanged, to give

$$u = \frac{1}{a}(1 + \epsilon g(\theta) + O(\epsilon^2)),$$

where $\epsilon \ll 1$. Using a Taylor series expansion for $f(1/u)$ and collecting powers of ϵ, show that the path equation reduces to

$$\left(\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2} + \left(3 + a\frac{f'(a)}{f(a)}\right) g\right) \epsilon = O(\epsilon^2) .$$

(iii) Hence, show that if $f(r) = -\gamma r^{-\lambda}$, where γ and λ are constants, a perturbation to a circular orbit can grow unboundedly for increasing θ if $\lambda > 3$.

[13 marks]

END OF EXAMINATION PAPER