MATH10222, Calculus & Applications : Examples 5

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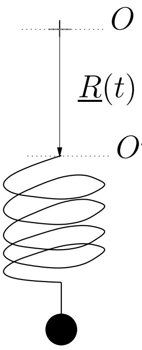
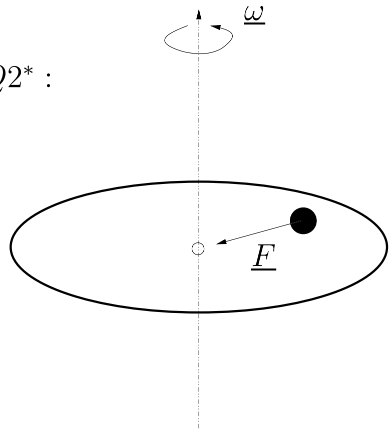
Questions marked with an asterisk are to be handed in to your supervisor.

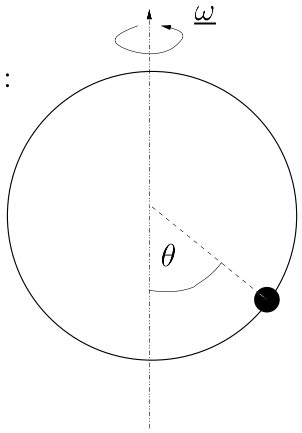
$Q1^*$:



$Q2^*$:



$Q3$:



Q1*. [Chapter 4, section 1] A particle P of mass m is attached to a point O' by a light spring of natural length l. P hangs vertically below the origin O' in a uniform gravitational field of strength $-g\underline{k}$, where $g > 0$ is a constant. The position of P relative to O' is denoted by $-(l + \zeta(t))\underline{k}$ where $|\zeta| < l$ is the extension/compression of the spring. The force exerted on the particle by the spring is $\gamma\zeta(t)\underline{k}$, where $\gamma > 0$ is a constant.

The attachment point O' has a position vector $\underline{R}(t)$ relative to an origin O, which is fixed in an inertial frame of reference.

(i) If $\underline{R}(t) = \underline{0}$ for all time t (i.e., O' and O are the same point), then show that the equation of motion for the particle P is

$$\ddot{\zeta} + \frac{\gamma}{m}\zeta = g\,.$$

(ii) Solve the above equation for $\zeta(t)$. What value of ζ corresponds to an equilibrium solution?

(iii) Show that the equation of motion above still applies to the case of $\underline{R} = Ut\underline{k}$, with any constant U.

(iv) Now consider the case of $\underline{R} = \cos(\omega t)\underline{k}$, which corresponds to oscillating (up and down) the point to which the spring is attached. In this case, derive the corresponding equation for ζ. Show that the equation is as derived for (i) above, but with g replaced by $\hat{g}(t)$, where $\hat{g}(t) = g - \omega^2\cos(\omega t)$. (i.e., oscillating the support point is equivalent to having a time-varying gravitational acceleration.)

(v) Solve this equation of motion for $\zeta(t)$ when $\omega^2 \neq \gamma/m$.

Hint: You should compare this question with Q1 of Examples 5 from Matthias Heil's part of the course.

Q2*. [Chapter 4, section 5]

A particle, P, of mass m is placed on a rotating turntable that spins relative to an inertial frame with an angular frequency vector of $\underline{\omega} = \omega \underline{k}$. Here $\underline{k}$ is perpendicular to the turntable.

The motion of P on the turntable is described in a Cartesian coordinate system $\{\underline{i}, \underline{j}, \underline{k}\}$ that rotates *with* the turntable. The position of P is given by $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j}$ where $\underline{i}$ and $\underline{j}$ are in the plane of the turntable. You are given that, in this non-inertial frame of reference, the governing equation is

$$m\ddot{\underline{r}} = \underline{F} - 2m\underline{\omega} \wedge \dot{\underline{r}} - m\underline{\omega} \wedge (\underline{\omega} \wedge \underline{r}) \,, \tag{1}$$

in the usual notation.

(i) Simplify the second and third terms on the right of equation (1).

(ii) The turntable is designed so that P is attracted to the centre of rotation by a force $\underline{F}$, where

$$\underline{F} = -m\omega^2 x \, \underline{i} - m\omega^2 y \, \underline{j} \,;$$

this is the only force acting on P. Show that the position of P satisfies

$$\ddot{x} = 2\omega \dot{y} \,,$$

$$\ddot{y} = -2\omega \dot{x} \,.$$

(iii) Combine the two equations for x and y into a single complex equation for $\psi = x + iy$. Solve this complex equation to show that P will, in general, oscillate in circles relative to the turntable at frequency 2ω.

Q3. [Chapter 4, sections 5] [HARD(ER!)]

A particle P of mass m is constrained to move on a smooth, circular wire hoop of radius a placed in a uniform gravitational field of strength g. The hoop rotates uniformly, about a diameter that is **vertical**, with an angular frequency ω. The vector equation of motion in the frame of reference that rotates with the hoop is given by equation (1) above.

(i) Show that

$$ma\ddot{\theta}(t) = -mg \sin\theta(t) + \omega^2 ma \sin\theta(t) \cos\theta(t) \,,$$

where θ is as shown in the diagram. Is the Coriolis force important in this problem?

Hint: The angular frequency vector is $\underline{\omega} = -\omega \cos\theta \, \hat{\underline{r}} + \omega \sin\theta \, \hat{\underline{\theta}}$ in the usual notation.

(ii) Show that if $\omega^2 > g/a$ then there are four values of θ at which the particle may remain at rest relative to the wire as it rotates.

Hint: two of the positions are obvious!

(iii) **Harder:** Now examine the linear stability of the four equilibrium solutions by deriving an equation for small perturbations to the position. To do this write:

$$\theta(t) = \alpha + \epsilon \psi(t) \,,$$

where α denotes any of the four equilibrium positions as determined in (ii) above. By neglecting any terms of $O(\epsilon^2)$, show that

$$a\ddot{\psi}(t) = \left(\omega^2 a \cos 2\alpha - g \cos\alpha\right) \psi(t).$$

Now find the solution for $\psi(t)$ at each of the four values of α. At which values of α can you show that $|\psi(t)|$ remains bounded for all t?

Hint: We used the same approach in Chapter 2, section 4.2 in the notes, albeit in a simpler context.

(iv) Can you say at which points you would expect to see the particle if you did the 'experiment' for increasing values of ω, starting with a non-rotating wire ($\omega = 0$) and then gradually increasing the rate of rotation?

You should try the questions without looking here first ... but some 'answers' are as follows :

Q1: (ii) The equilibrium point is at $\zeta = \zeta_e$, where

$$\zeta_e = \frac{mg}{\gamma} \, .$$

(v) The solution for $\omega^2 \neq \gamma/m$ is

$$\zeta(t) = \zeta_e + \frac{m\omega^2}{m\omega^2 - \gamma} \cos(\omega t) + A\cos(\sqrt{\gamma/m}\, t) + B\sin(\sqrt{\gamma/m}\, t) \, ,$$

for constants A and B.

Q2:

$$x + iy = A(\cos(2\omega t) - i\sin(2\omega t)) + B$$

for (complex) constants A and B.

Q3: (ii) The four positions are $\theta = 0, \pi, \alpha, -\alpha$ where

$$\alpha = \arccos\left(\frac{g}{\omega^2 a}\right) \, .$$

Aside: Question 2 has a serious point. You can make such a table/force-field by putting gravity back into the problem and making the table shape a paraboloid of revolution. The shape is then chosen so that the component of the particle's weight tangential to the table's surface exactly balances the centrifugal force for a given value of ω. On a table rotating at this rate ω, a particle will then perform pure "inertial oscillations" at the "Coriolis frequency" 2ω, being 'free' from other forces.

If you don't believe the mathematics, see:

`http://hewitt.ddns.net/files/MATH10222/movie.mpg`[1]

(or movie.mov for an alternative format) for an example movie of a 'particle' moving on just such a table. Two views of the SAME motion are shown in the movie, corresponding to the inertial (room) and non-inertial (rotating-table) frames of reference (on the left/right respectively). The oscillations demonstrated in this very simple example turn out to be important in the fluid dynamics of the Earth's oceans/atmosphere.

Full solutions will be provided after the problems have been discussed in the supervision classes.

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[1] I'm not sure who did this experiment, otherwise I'd link to the source here!