MATH10222, Calculus & Applications : Examples 4

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Questions marked with an asterisk are to be handed in to your supervisor.

Q1*. [Chapter 1; Example 1.1; Chapter 2, section 2] A particle P is released from rest at an origin O and moves in a uniform gravitational field. If the distance *downwards* from the origin is z, show that

$$\ddot{z} = g\,,$$

where g is the local acceleration due to gravity.

(i) Solve for $z(t)$.

(ii) Answer the following problem using your solution for $z(t)$ with $g \approx 9.81\,m/s^2$:

You're sitting in a 10th floor office. Whilst staring aimlessly out of the window you hear a scream, "I hate Microsoft!", and see a laptop plummet past to the pavement below. In a moment of boredom you denote the distance from the bottom to the top of your window as h, and the time that the laptop took to cross that distance as τ.

You estimate that $h = 1m$, $\tau = 0.05s$. What floor do you think the laptop was dropped from, if each floor is approximately $4m$ and the laptop is released from rest?

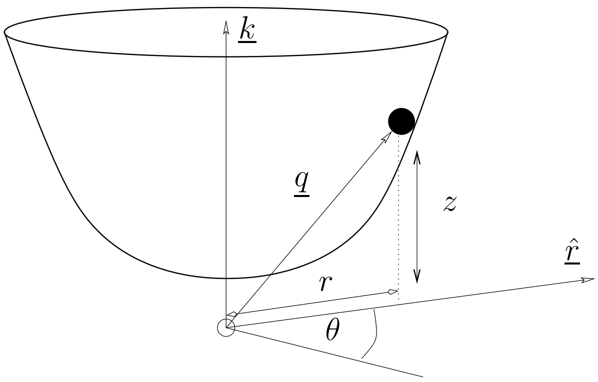

Figure 1: A diagram for question 2.

Q2*. [Chapter 2, Example 8.1] A particle P of mass m and position vector $\underline{q}$ is moving in contact with the smooth inside of a body of revolution as shown in the figure ("smooth" indicates that the only forces in this case are the normal (perpendicular) reaction force and the particle's weight). The axis of symmetry of the surface of revolution, $\underline{k}$, is vertical.

(i) Draw a diagram showing the particle in the surface, and the forces (the weight and normal reaction) acting on P. Hence explain why $\underline{L}_o \cdot \underline{k} = 0$, where $\underline{L}_o = \underline{q} \wedge \underline{F}$ (the moment), and $\underline{F}$ is the resultant force acting on P.

(ii) Using the angular momentum equation

$$\frac{\mathrm{d}\underline{H}_o}{\mathrm{d}t} = \underline{L}_o \,,$$

show that

$$\underline{H}_o \cdot \underline{k} = h \quad \text{(a constant)} \,. \tag{1}$$

(iii) Given that the position vector of P can be written as $\underline{q} = r\hat{\underline{r}} + z\underline{k}$ in a coordinate system relative to O (as shown in the figure), with velocity $\dot{\underline{q}} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}} + \dot{z}\underline{k}$, and $\underline{H}_o = \underline{q} \wedge m\dot{\underline{q}}$, show that (1) reduces to

$$mr^2\dot{\theta} = h \,.$$

Hint: In chapter 2, section 6 we did some very similar steps. The only (slight) difference here is that P is not confined to move in a plane, it is moving on a surface, so its position has an **extra** vertical component $z(t)\underline{k}$.

Q3. [Chapter 3, section 4] For a satellite of unit mass in an elliptical orbit of the Earth, conservation of energy provides the constraint

$$\frac{1}{2}[U(r)]^2 - \frac{MG}{r} = -\frac{MG}{2a} \,,$$

where $U(r)$ is the satellite's speed at a distance r from the centre of the Earth, M is the mass of the Earth, G is the gravitational constant and a is the semi-major axis of the ellipse.

(i) Suppose that the satellite is in a circular orbit of radius $r = d$ with constant speed $U(r) = u_0$. Find u_0 in terms of M, G and d.

(ii) At some point during the motion, a booster rocket is fired that impulsively changes the speed of the satellite from u_0 to $k\,u_0$ (in the direction of travel), where $k > 0$ is a constant. Find the value of k that would move the satellite into an elliptical orbit of semi-major axis $3d/2$.

(iii) When the satellite is at $r = 2d$, a second-stage booster is fired that impulsively increases the speed from $U(r) = u_1$ to μu_1 (in the direction of travel), where $\mu > 0$ is a constant. Determine u_1, then compute the value μ required to leave the satellite in another circular orbit of radius $2d$.

Q4*. [Chapter 2, examples 7.1, 7.2] A particle P of mass m moves in the central field of force

$$\underline{F} = mf(r)\hat{\underline{r}}\,,$$

where $f(r) = -\gamma^2/r^3$ and γ is a positive constant.

As shown in the lecture notes, the "path equation" is

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = -\frac{1}{H_0^2 u^2} f(1/u)\,,$$

where $u = 1/r$ and H_0 is a constant. For the case $H_0 = 15\gamma/\sqrt{209}$ solve for $u(\theta)$.

Q5. [see Q2 above] [HARD(ER)!] A particle P has mass m and slides in contact with the smooth inside of a circular cone of semi-angle α (i.e., this is the same situation described in Q2 above, but for a surface that is a downward pointing cone).

Initially P is a distance a from O (the vertex of the cone), when it is projected horizontally along the inside surface of the cone with speed u. Therefore, at $t = 0$, we have $|\underline{q}| = a$ and $\dot{\underline{q}} = u\,\hat{\underline{\theta}}$, where $\underline{q}$ is as defined in Q2.

(i) Use these initial conditions to show that the constant h, as defined in Q2, is $h = mau\sin\alpha$. Hence show that

$$q^2 \sin\alpha\,\dot{\theta} = au\,,$$

where $q = |\underline{q}|$.

(ii) If the force acting on the particle is

$$\underline{F} = -mg\underline{k} + \underline{N}\,,$$

where $\underline{N}$ is the normal reaction force from the conical surface, explain (either by a brief sentence, or mathematically) why

$$\underline{F} \cdot \dot{\underline{q}} = -mg\dot{z}\,,$$

(i.e., the normal reaction does not contribute to the work done integral).

(iii) Hence show that a potential function for this motion is $V = mgz$, where

$$V = -\int \underline{F} \cdot \dot{\underline{q}}\,\mathrm{d}t\,,$$

(i.e., although there is a normal reaction force acting on P, we don't have to worry about it and just have the usual potential for a uniform gravitational field).

(iv) Using this potential, show that conservation of energy (i.e., $T + V = E$, with $T = \frac{m}{2}|\dot{\underline{q}}|^2$, in our usual notation) leads to

$$\dot{q}^2 = (q-a)\left(\frac{u^2(q+a)}{q^2} - 2g\cos\alpha\right)\,.$$

(v) Find the maximum/minimum distance of P from the apex of the cone.

You should try the questions without looking here first ... but some 'answers' are as follows :

Q1: from the 15^{th} floor.

Q2: All the answers are in the question.

Q3: $k = \sqrt{4/3}$, $\mu = \sqrt{3/2}$.

Q4: $u = \alpha \cos \frac{4\theta}{15} + \beta \sin \frac{4\theta}{15}$, where α and β are constants. (We can't determine them since the question gives no initial conditions for the particle!).

Q5: $q = a$ and $q = \frac{u^2 + u\sqrt{u^2 + 8ga\cos\alpha}}{4g\cos\alpha}$ are the points where $\dot{q} = 0$.

Full solutions will be provided after the problems have been discussed in the supervision classes.
