# MATH10222, Calculus & Applications : Examples 3

Dr. R.E. Hewitt, http://hewitt.ddns.net
Last modified: 2016-03-17

---

**Questions marked with an asterisk are to be handed in to your supervisor.**

**Q1. [Chapter 1, example 5.1]** Suppose a particle is moving with a constant *speed* $U$. By considering the dot-product $\underline{\dot{r}}\cdot\underline{\dot{r}}$, show that any acceleration must always be perpendicular to the velocity.

**Hint:** This is actually simple (once you spot how to do it!) and takes just a few lines.

**Q2*. [Examples 2, Q3]** A particle $P$ of constant mass $m$ has a position vector

$$\underline{r} = x(t)\underline{i} + y(t)\underline{j}\,,$$

(in the usual notation) and moves in a uniform gravitational field $-g\underline{j}$. At time $t = 0$, $P$ is at the origin of the coordinate system and is projected with a speed $U$ at an angle $0 \le \theta \le \pi$ to the vector $\underline{i}$.

(i) Use Newton's second law to show that $\underline{\ddot{r}} = -g\underline{j}$. Hence show that $y = 0$ at a time $t = \tau > 0$ where

$$\tau = \frac{2U\sin\theta}{g}\,.$$

(ii) At time $t = \tau$, find the displacement of the particle from the origin of the coordinate system.

(iii) Find the time at which $P$ reaches its maximum height above $y = 0$, and determine what that maximum height is.

(iv) For a fixed $U$, what projection angle will lead to a maximum displacement of $P$ from the origin when $t = \tau$.

**Hint:** There is no mention of air resistance here, so you should assume that the only force acting on $P$ during its motion arises from the uniform gravitational field. The question is not very different from Ex 2, Q3.

**Q3***. **[Chapter 2, section 4; examples 4.1, 4.2 & 4.3]** A particle $P$ of unit mass moves along the positive $x$-axis under the influence of a force

$$F(x) = \frac{16}{x^2} - 2\,, \quad x > 0\,.$$

(i) Sketch the shape of a potential function $V(x)$.

(ii) Initially $P$ is released from rest at $x = 1$. Show that in the subsequent motion, the particle position is bounded between $x = 1$ and $x = 8$.

(iii) Express the time taken for the particle to travel from $x = 1$ to $x = 8$ as a definite integral with respect to $x$.

(iv) Show that there is a single equilibrium position for $P$ and find the energy associated with this equilibrium state.

**Hint:** See the examples in the lecture notes!

**Q4.** **[Chapter 2, section 4]** A particle $P$ of mass $m$ moves under the gravitational attraction of a mass $M$ that is fixed at an origin $O$. Initially, $P$ is released from rest at a distance $a$ from $O$. Since the force acts radially, and the initial velocity is zero, the particle moves along a line with

$$m\ddot{r} = -\frac{M\,m\,G}{r^2} \tag{1}$$

where $r$ is the distance $OP$, and $G$ is the universal gravitational constant.

By *deriving* the energy equation for this problem show that in the subsequent motion

$$v^2 = 2MG\left(\frac{1}{r} - \frac{1}{a}\right)\,,$$

where $v = \dot{r}$ is the particle's velocity.

By integrating the expression for $v$ with respect to time, find the time taken for $P$ to reach $O$.

**Hints:**

**(i)** If you can't remember how to get the energy equation, look how it is derived in the notes. It is the same process here, multiply (1) by $\dot{r}$ and integrate with respect to time.

**(ii)** You may use the following integral without proof,

$$\int_0^1 \sqrt{\frac{s}{1-s}}\,\mathrm{d}s = \frac{\pi}{2}\,.$$

**(iii)** This may look like motion in a plane at first sight, but the initial conditions are such that the motion is in fact along a line!

**You should try the questions without looking here first ... but some 'answers' are as follows :**

Q2: The displacement is

$$\frac{U^2 \sin 2\theta}{g} \underline{i} \, .$$

The time at which the maximum height is reached is $\tau/2$ and the maximum height is

$$\frac{U^2 \sin^2 \theta}{2g} \, .$$

Greatest displacement (in the $\underline{i}$ direction) occurs for $\theta = \pi/4$.

Q4: If we denote the time taken by $\tau$, then

$$\tau = \frac{\pi}{2} \left( \frac{a^3}{2MG} \right)^{1/2} \, .$$

---

**Full solutions will be provided after the problems have been discussed in the supervision classes.**

---