MATH10222, Calculus & Applications : Examples 2

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Questions marked with an asterisk are to be handed in to your supervisor.

Q1. [Chapter 1, sections 3 & 4] In a Cartesian coordinate system with basis vectors $\{\underline{i}, \underline{j}, \underline{k}\}$, a body is acted upon by three forces:

$$\underline{F}_1 = 2\underline{i} + 4\underline{j} + 3\underline{k}, \quad \underline{F}_2 = -3\underline{i} + 5\underline{j} - 3\underline{k}, \quad \underline{F}_3 = \underline{i} - 9\underline{j}\,.$$

The force $\underline{F}_i$ acts through a position $\underline{r}_i$, where

$$\underline{r}_1 = \underline{i} + \underline{j} + \underline{k}\,, \quad \underline{r}_2 = 2\underline{j} - \underline{k}\,, \quad \underline{r}_3 = -\underline{i} + 2\underline{k}\,.$$

Find the resultant force acting on the body and the total moment of the forces about the origin O. Now compute the total moment of the forces about the position $\underline{r}_1$. Is the system in equilibrium?

Hint: We defined the 'resultant' and 'moment' in the lecture notes and this should be (very) easy! To determine the moment about the position $\underline{r}_1$, you need the position vectors of the points $\underline{r}_2$ and $\underline{r}_3$ *relative* to $\underline{r}_1$.

Q2. [Chapter 1, section 7; Example sheet 1, question 4] The European Space Agency recently launched "Cryosat" [1], a satellite that aimed to monitor the thickness of the polar ice caps. The satellite was designed to orbit in a circular path at low altitude, $720km$, above the mean radius of the Earth ($6370km$). To the nearest minute, how long would it take for Cryosat to orbit the Earth once?

(You will require the universal gravitational constant $G \approx 6.67 \times 10^{-11}\, m^3 kg^{-1} s^{-2}$ and the mass of the Earth $M_E \approx 5.97 \times 10^{24}\, kg$.)

Hint: Newton's law of gravitation tells you the force acting on the satellite. Newton's 2nd law of motion relates this force to the acceleration of the satellite. Ex 1, Q4 relates the frequency of rotation ω to the acceleration. Linking these together means you can determine the frequency, then the time taken to rotate by 2π radians is $2\pi/\omega$ seconds. (You should not need to know the mass of the satellite!)

[1] This question was written in the 90 minutes between Cryosat's "successful" launch and first reports of it subsequently crashing with a £90M splash in the Arctic Ocean! Fortunately, Cryosat-2 did rather better.

Q3*. [Chapter 1, examples 2.3, 6.2; Chapter 2, example 3.1] A particle of mass m moves along a smooth (ie, *frictionless*) solid plane inclined at an angle, $0 < \alpha < \pi/2$, to the unit vector $\underline{i}$, in a uniform gravitational field $-g\underline{j}$.

(i) Draw a diagram of the particle and plane, showing the forces acting on the particle. (See Chapter 1, example 6.2)

(ii) Resolve the weight of the particle into two components, one parallel and one perpendicular to the plane. (See Chapter 1, example 2.3)

(iii) If the particle is confined to *stay on the plane*, show that Newton's second law reduces to the scalar ODE:

$$\ddot{X}(t) = -g \sin \alpha \,,$$

where $X(t)$ is the position of the particle measured up the plane.

(iv) The particle is set in motion at $X = 0$ (on the plane) at $t = 0$, and fired with a speed U up the slope of the plane. By applying these initial conditions, solve the ODE for $X(t)$.

(v) Find the maximum distance travelled up the plane and the time at which this maximum is reached.

Hint: At the maximum distance up the plane, the particle's velocity is (instantaneously) zero.

Q4*. [Chapter 2, section 3, example 3.2] Consider the configuration of Q3 again, but this time assume that there is also a *resistive* force between the plane and the particle that acts parallel to the plane and is proportional to the *velocity* of the particle (with constant of proportionality $m\gamma > 0$). The governing ODE in this case is

$$\ddot{X}(t) = -g \sin \alpha - \gamma \dot{X}(t) \,.$$

Using the same initial conditions that were applied in Q3, determine the maximum distance travelled up the plane. At what time is this maximum distance reached?

Show that considering the limit of $\gamma \to 0$ in your answers for the maximum distance and time taken, you can recover the results of Q3 above.

Hint: The final part is more subtle than simply setting $\gamma = 0$ in the answers.

Q5. [Chapter 1, section 7; Chapter 2, section 1] A particle of mass m is projected upwards with speed V from the surface of the Earth. You are given that the displacement $x(t)$, relative to the Earth's surface, satisfies

$$m\ddot{x}(t) = -\frac{mEG}{(R + x(t))^2} \,,$$

where E is the mass of the Earth, R is the radius of the earth and G is the universal gravitational constant.

By making the substitution $x(t) = \xi(\tau)V^2/g$ where $t = \tau V/g$ and $g = EG/R^2$, show that

$$\ddot{\xi}(\tau) + \frac{1}{(1 + \epsilon\, \xi(\tau))^2} = 0 \,,$$

subject to $\xi(0) = 0$ and $\dot{\xi}(0) = 1$, with $\epsilon = V^2/(Rg)$.

[This is the same equation discussed in example sheet 6 from the first part of the course, in which you assume ϵ is small.]

You should try the questions without looking here first ... but some 'answers' are as follows :

Q1: $\underline{L}_o = 16\underline{i} + 4\underline{j} + 17\underline{k}$.

Q2: 99 minutes.

Q3: (v) the time taken is

$$T = U/(g \sin\alpha)\,,$$

and the maximum distance is

$$x_{max} = U^2/(2g\sin\alpha)\,.$$

Q4: the time taken is

$$T = -\frac{1}{\gamma}\ln\left(\frac{g\sin\alpha}{g\sin\alpha + \gamma U}\right)\,,$$

and the maximum distance is

$$x_{max} = \frac{U - g\sin\alpha T}{\gamma}\,.$$

Full solutions will be provided after the problems have been discussed in the supervision classes.
