MATH10222, Chapter 3: Planetary Motion

Dr. R.E. Hewitt, http://hewitt.ddns.net
Last modified: 2017-04-28

We will run through the material of sections 1-3 very quickly in the lecture to show how the results of Chapter 2 can be applied to planetary motion. However, I will not ask any questions regarding the proof of Kepler's 2nd or 3rd laws in the final examination (they are marked as 'extra' below).

1 Kepler

Extra

Johannes Kepler (1571 – 1630) was a German mathematician astronomer who, between 1596 and 1619 worked on planetary motion. He took the observational data of Tycho Brahe (1546 – 1601) and after considerable effort produced three 'laws of planetary motion' – based purely on Brahe's numerical data. Kepler's laws are:

1. Relative to the Sun, the planets describe elliptical paths with the Sun at one focus.
2. The line joining the Sun to a planet sweeps out an equal area in equal times.
3. The time required for a planet to orbit the Sun (i.e., the period of its motion) is proportional to the semi-major axis of the ellipse raised to the power 3/2. The constant of proportionality is the same for all planets.

2 Revision: Properties of an ellipse

In Cartesian coordinates $\{x, y\}$, an ellipse is defined by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a \geq b$ is the semi-major axis and b the semi-minor axis.

In plane-polar coordinates $\{r, \theta\}$ (centred at a focus), the same ellipse is defined by

$$\frac{1}{r} = \frac{1}{l}\left(1 - e\cos\theta\right),$$

where $e^2 = 1 - b^2/a^2$ and $l = b^2/a$.

Further revision on the properties of an ellipse can be found at:

`http://en.wikipedia.org/wiki/Ellipse`

3 From Kepler to Newton (& back!)

Kepler based his 'laws' on observational data, but Newton showed that (eventually) calculus[1], his laws of motion and his law of gravitation led to Kepler's three statements.

Recall that in Chapter 1, section 7 we noted that Newton's law of gravitation for two bodies provides a force of magnitude

$$\frac{Gm_1m_2}{|\underline{r}|^2},$$

directed towards the other body. If $m_1 = M$ (the mass of the Sun) and $m_2 = m$ (the mass of the Earth), then $M >> m$ and we can approximate the mass m_1 (i.e., the Sun) to be stationary.

Therefore, gravity due to the Sun may be considered to be a central field of force with

$$\underline{F} = -\frac{mMG}{|\underline{r}|^2}\hat{\underline{r}},$$

where $\underline{r}$ is the position of the Earth measured relative to the (now fixed position of the) Sun.

In the notation of Chapter 2, section 6, we introduce a scalar force $f(r)$ such that $\underline{F} = mf(r)\hat{\underline{r}}$, with

$$f(r) = -\frac{MG}{r^2},$$

where $|\underline{r}| = r$. To simplify further we will define a constant $\gamma = MG$.

As in all our other central fields of force problems we have Newton's 2nd law:

$$\ddot{\underline{r}} = f(r)\hat{\underline{r}},$$

and all the results that followed from this.

3.1 Kepler I : elliptical paths

Recall that the *path equation* is

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{f(1/u)}{H_o^2u^2},$$

where $u = 1/r$ and $H_o = r^2\dot{\theta}$.

In this case, $f(r) = -\gamma/r^2$ so that the path equation is

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{\gamma}{H_o^2},$$

a linear, inhomogeneous, constant coefficient ODE for $u(\theta)$.

As in part 1 of the course (with Matthias Heil) the solution can be written as

$$u = A\cos\theta + B\sin\theta + \gamma/H_o^2,$$

[1]Newton's original discussion was more geometric, only later was it discussed in terms of his calculus.

for constants A and B. Equivalently we can write

$$u = \frac{\gamma}{H_o^2} - C\cos(\theta - \alpha)\,,$$

for constants C and α (which can be related to A and B).

Using $u = 1/r$ yields:

$$\frac{1}{r} = \frac{\gamma}{H_o^2}\left(1 - e\cos(\theta - \alpha)\right)\,,$$

where $e\gamma/H_o^2 = C$ is a constant.

We note that (as Kepler claimed) the path is an ellipse of eccentricity e (this is precisely the plane-polar form of an ellipse given above just rotated by an angle α, and $l = H_o^2/\gamma$).

Extra

3.2 Kepler II : area swept out

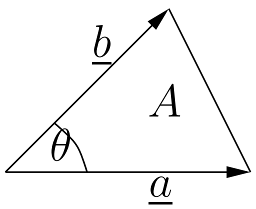
Let's consider two vectors $\underline{a}$ and $\underline{b}$ in a plane, whose angle of separation is θ. The area spanned by these two vectors is

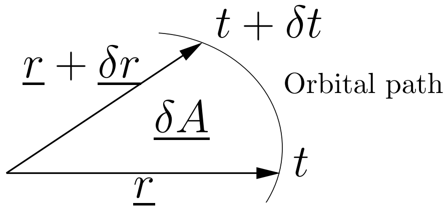
$$A = \frac{1}{2}\,|\underline{a}|\,|\underline{b}|\sin\theta,$$

so

$$A = \frac{1}{2}\,|\underline{a}\wedge\underline{b}| = |\underline{A}|\,,$$

say, where $\underline{A} = (\underline{a}\wedge\underline{b})/2$.





In a time δt a planet sweeps out a surface $\underline{\delta A}$. At time t the planet's position is $\underline{r}$ and at time $t + \delta t$ it is $\underline{r} + \underline{\delta r}$. As above, we know that

$$\underline{\delta A} = \frac{1}{2}\,\underline{r}\wedge\underline{\delta r}\,,$$

so that the rate of change of area is

$$\frac{\underline{\delta A}}{\delta t} = \frac{1}{2}\,\underline{r}\wedge\frac{\underline{\delta r}}{\delta t}\,.$$

As $\delta t \to 0$ we are left with

$$\frac{\mathrm{d}\underline{A}}{\mathrm{d}t} = \frac{1}{2}\,\underline{r}\wedge\underline{\dot{r}}\,.$$

Recall that we know, in a central field of force, $\underline{r}\wedge\underline{\dot{r}}$ is a constant vector and the motion is confined to a plane. Thus the rate of change of area swept out by the planet is constant, that is, an equal area is 'swept out' by the planet in equal time intervals, as Kepler claimed.

3.3 Kepler III : energy/period of an orbit

As in chapter 2, examples 6.1, 6.2 we can define a potential for this central field of force:

$$V(r) = -\int f(r)\mathrm{d}r = \int \frac{\gamma}{r^2}\mathrm{d}r = -\gamma r^{-1} + V_o\,,$$

for some constant of integration V_o. As usual for a potential function we can choose the constant to be something convenient, in this case we can choose $V_o = 0$, so that $V(r) = -\gamma/r$.

If the planet has a velocity U_o at its furthest point from the Sun, then the angular momentum at that point is determined from

$$H_o = r^2\dot{\theta}\,,$$

with $r = a(1+e)$ (the maximum distance from a focus in an ellipse) and $r\dot{\theta} = U_o$. Therefore $H_o = U_o a(1+e)$, where e is the eccentricity of the ellipse.

We can also use the energy conservation equation:

$$\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = E\,,$$

where at the furthest point again, $\dot{r} = 0$, $r\dot{\theta} = U_o = H_o/(a(1+e))$ and $V = -\gamma/(a(1+e))$. So we find that

$$E = \frac{1}{2}\,\frac{H_o^2}{a^2(1+e)^2} - \frac{\gamma}{a(1+e)}\,.$$

We can simplify further by noting from Kepler I that $l = H_o^2/\gamma$ and for an ellipse $l = a(1-e^2)$, thus

$$E = -\frac{\gamma}{2a} = -\frac{MG}{2a}\,. \qquad (1)$$

We know from Kepler II that

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}H_o\,,$$

and therefore

$$\frac{\mathrm{d}A}{\mathrm{d}\theta}\,\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}H_o\,.$$

For the planet to orbit once, θ increases by 2π whilst t increases by T (the period). Thus integrating yields

$$\int_0^{2\pi} \frac{\mathrm{d}A}{\mathrm{d}\theta}\,\mathrm{d}\theta = \int_0^T \frac{1}{2}\,H_o\,\mathrm{d}t\,.$$

The LHS of this expression is the area of an ellipse (πab), so we find that

$$\pi a\,b = \frac{1}{2}\,H_o T\,.$$

As before, we know that $H_o^2 = \gamma l$, $l = b^2/a$ and thus $H_o^2 = \gamma^2 b^2/a$, thus

$$T = \frac{2\pi ab}{H_o} = \frac{2\pi}{\sqrt{\gamma}}a^{3/2}\,.$$

As Kepler claimed, the period of a planet's orbit scales like the semi-major axis to the power of 3/2 and the constant of proportionality is the same for each planet.

4 Orbital transfer

In the previous section we showed that a particle P in an elliptical orbit of semi-major axis a (or equivalently major axis $2a$) has an energy of

$$E = -\frac{MG}{2a}\,. \tag{2}$$

Therefore, as in Chapter 2, the conservation of energy equation is

$$\frac{1}{2}[U(r)]^2 - \frac{MG}{r} = E = -\frac{MG}{2a}\,, \tag{3}$$

where we have used $U(r)$ to denote the speed of P at a radius r.

Using just this information we can now work out how to move a satellite from one circular orbit to another. We'll do this by considering a specific example.

4.1 Example: Orbital transfer

Consider a particle P in a circular orbit of radius d. We can move this particle to another circular orbit of radius $4d$ by using a two-stage process. Each of the two stages involve an increase in speed of the particle that changes the orbit energy.

Question: Find by what factor the speed of P must be increased in each of the two booster stages.

Answer: Let's take the two stages in turn.

Stage 1: The boost to a transfer orbit

The initial orbit is a circle of radius d. The 'major axis' (ie. diameter in this case because it is circular) of this orbit is $2d$ in (2), so the energy of this orbit is

$$E = E_{initial} = -\frac{MG}{2d}\,.$$

Therefore the conservation of energy equation (3) reduces to

$$\frac{1}{2}[U_0]^2 - \frac{MG}{d} = E_{initial}\,, \tag{4}$$

when evaluated at the point A ($r = d$) in figure 1, where U_0 is the speed. Hence

$$U_0^2 = \frac{MG}{d}\,.$$

As this orbit is a circle, the particle has this same speed at all points on the orbit.

For the transfer orbit, we boost the speed of P from $U_0 \to kU_0$ in order to change the energy level $E_{initial} \to E_{transfer}$. Here we choose E_1 to be the energy of the transfer orbit, which is an orbit of major-axis $5d$, so (2) provides

$$E = E_{transfer} = -\frac{MG}{5d}\,.$$

Again, evaluating (3) at the point A after the speed increase, we find

$$\frac{1}{2}[kU_0]^2 - \frac{MG}{d} = E_{transfer}\,,$$

so

$$k^2U_0^2 = \frac{8MG}{5} \, ,$$

and therefore

$$k = \sqrt{\frac{8}{5}} \, .$$

Stage 2: The boost to a final orbit

On the transfer orbit, we now move to a maximum radius of $4d$, but this orbit is elliptical. Because we want a final orbit that is circular, we need to increase the speed again. So at point B ($r = 4d$) we increase from $U_1 \to \mu U_1$ with the aim of achieving a new energy level of

$$E = E_{final} = -\frac{MG}{8d} \, ,$$

which is the energy required for a circle of major-axis (ie., diameter) $8d$. Here U_1 is the speed at point B (at $r = 4d$) on the transfer orbit, which (3) shows to be

$$U_1^2 = 2\left(E_{transfer} + \frac{MG}{4d}\right) = \frac{MG}{10d} \, .$$

To find μ we simply apply (3) again, after the speed/energy has been increased, which gives

$$\frac{1}{2}[\mu U_1]^2 - \frac{MG}{4d} = E_{final} = -\frac{MG}{8d} \, .$$

So

$$\mu^2 U_1^2 = 2\left(\frac{MG}{4d} - \frac{MG}{8d}\right) = \frac{MG}{4d} \, ,$$

and then

$$\mu = \sqrt{\frac{5}{2}} \, .$$

So to move from a circular orbit of radius of d to a new circular orbit of radius $4d$ requires two speed boosts (in the direction of travel), the first being a relative increase of $\sqrt{8/5}$. We then wait until the particle is at its maximum distance from the origin ($4d$) before firing the second booster, which provides a relative increase of $\sqrt{5/2}$.

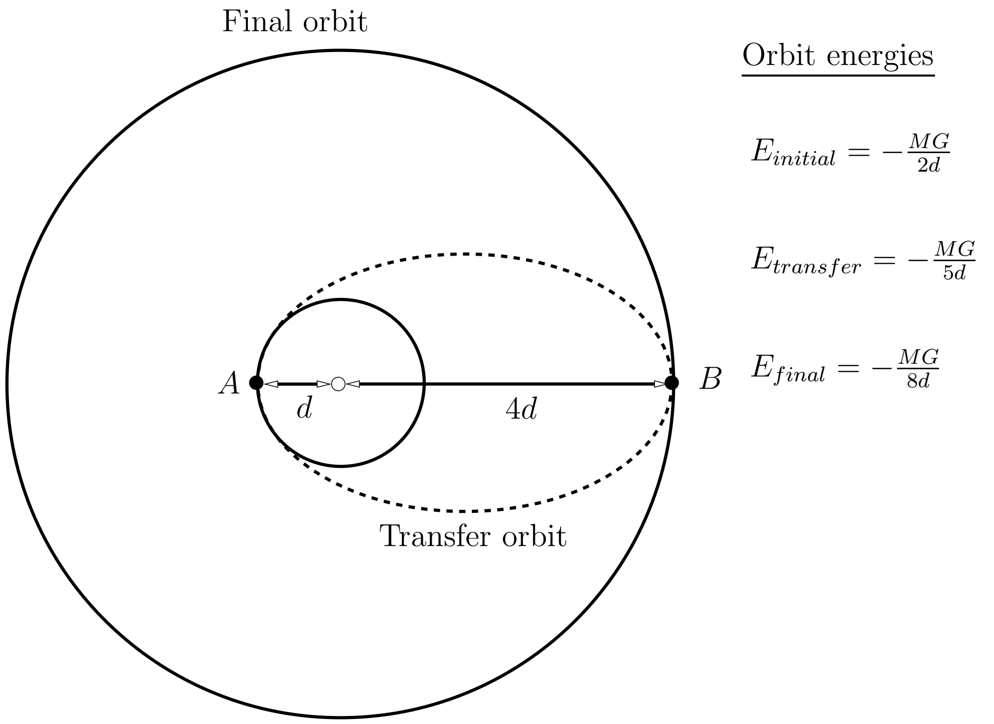


Figure 1: The orbital transfer example (4.1), with each of the three orbits shown. The energy associated with each orbit is also given, which simply follow from equation (2).